Received: *May 18, 1999*, Accepted: *June 15, 1999*

D-branes on $T^4/\mathbb{Z}_2$ and T-Duality

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Abstract: In this note we discuss D-branes on $T^4/\mathbb{Z}_2$ using the boundary states formalism. Explicit formulas for the untwisted boundary states inherited from the underlying T^4 and twisted states corresponding to branes wrapping collapsed 2-cycles at the orbifold singularities are given. The exact CFT description of the orbifold makes it possible to study how the boundary states transform under $R_i \mapsto \frac{1}{R_i}$ transformation on all directions of the underlying T^4. We compare their transformation law with results obtained from world volume considerations.

Keywords: Superstrings and Heterotic Strings, D-branes, String Duality, Conformal Field Models in String Theory.

JHEP06(1999)016

Contents

1. Introduction

Starting in [1], D-branes transverse to $\mathbb{C}^2/\mathbb{Z}_N$ orbifolds have been discussed extensively in the literature. Much less has been said about D-branes moving in curved compact backgrounds and on compact orbifold backgrounds, or even general CFT backgrounds. (See however [2] and [3, 4, 5] which discuss D-branes in Gepner models.) In this paper, we will study the toroidal orbifold $T^4/\mathbb{Z}_2$. This model has been studied from the point of view of world volume theories of D-branes in [6, 7, 8]. Here, a different point of view will be taken, namely, we will construct boundary states for the D-branes. This approach goes back to the work of Cardy in the context of rational conformal field theory [9]. Let us briefly describe the main ideas of this work.

The CFT description of open strings involves Riemann surfaces with boundaries as world sheets, e.g. the strip or the upper half plane. In the interior, the theory behaves like a theory on the full plane. The physics on the boundary depends on the boundary conditions imposed there. Consider a bulk theory realizing two isomorphic left and right-moving chiral algebras. On the boundary the left and right-moving generators must be related by an automorphism of the chiral algebra, otherwise the corresponding symmetry will not be preserved. In particular, since no energy is allowed to flow across the boundary, the operators $T(z)$ and $\tilde{T}(\bar{z})$ must be identified on the boundary.

These boundary conditions can be transformed to the closed string sector. In this language, the boundary conditions are implemented on boundary states, as given in [10]. However, for a consistent boundary state modular invariance imposes additional conditions. To see this, consider an open string one-loop amplitude, i.e. the world sheet is a cylinder. Alternatively, this diagram can be viewed as a closed string tree-level amplitude between boundary states. This leads to the condition, that the modular transformation of a transition amplitude between boundary states leads to an open string partition function, in particular, a sum of characters with integer coefficients (Cardy's condition).

The plan of this paper is as follows. In the next section we establish our conventions and notations by reviewing supersymmetric boundary states on tori. Our discussion will be brief and the reader is referred to [11, 12] for a more complete treatment. The twisted and untwisted orbifold boundary states are then constructed in section 3 where we also discuss their spacetime interpretation.

In section 4, we discuss T-duality on a $T^4/\mathbb{Z}_2$ orbifold. More precisely, we consider a transformation $\mathcal{T}$, which acts on the untwisted sector of the orbifold theory as the $R_i \mapsto 1/R_i$ transformation along all four circles of the underlying T^4. We point out that if $\mathcal{T}$ acts on the twist fields as a certain unitary transformation, it preserves the 3-point functions and is therefore a symmetry of the CFT. The action of $\mathcal{T}$ on the boundary states is then compared with a transformation given in [6] acting on the BPS vectors of the orbifold limit of type IIA on $K3$. We find agreement between the two. In the last section we discuss our results.

2. Boundary states in toroidal compactifications

Let us briefly review the construction of boundary states [13, 14] in a toroidal background where $x^6, \ldots, x^9$ are compactified with radii $R_6, \ldots, R_9$. To avoid introducing (super)ghosts, we will work in the light-cone gauge as in [15] and consider Dirichlet boundary conditions localizing the D-branes at the origin of all non-compact directions. The light-cone boundary conditions mean that we we will effectively describe $p+1$-instantons instead of p-branes, but these are related by a double Wick rotation. The problem at hand is constructing the NSNS and RR sector boundary states $|Dp;\eta;\vec{k}\rangle\!\rangle_{\substack{NSNS\\RR}}$ satisfying the boundary conditions

$$\begin{aligned}
(\alpha_n^i - \tilde{\alpha}_{-n}^i)|Dp;\eta;\mathbf{k}\rangle\!\rangle_{\substack{NSNS\\RR}} &= 0 \\
(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu)|Dp;\eta;\mathbf{k}\rangle\!\rangle_{\substack{NSNS\\RR}} &= 0\,, \qquad n \in \mathbb{Z} \\
(\psi_r^i - i\eta\tilde{\psi}_{-r}^i)|Dp;\eta;\mathbf{k}\rangle\!\rangle_{NSNS} &= 0 \\
(\psi_r^\mu + i\eta\tilde{\psi}_{-r}^\mu)|Dp;\eta;\mathbf{k}\rangle\!\rangle_{NSNS} &= 0\,, \qquad r \in \mathbb{Z} + \frac{1}{2} \\
(\psi_r^i - i\eta\tilde{\psi}_{-r}^i)|Dp;\eta;\mathbf{k}\rangle\!\rangle_{RR} &= 0 \\
(\psi_r^\mu + i\eta\tilde{\psi}_{-r}^\mu)|Dp;\eta;\mathbf{k}\rangle\!\rangle_{RR} &= 0\,, \qquad r \in \mathbb{Z}\,,
\end{aligned} \tag{2.1}$$

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where μ,ν the Neumann directions and i,j label the Dirichlet ones. The vector $\mathbf{k}$ denotes the continuous momenta carried by the bosonic Fock vacuum in the non-compact directions and the discrete momentum or winding quantum numbers in the compact directions $x^6,\ldots,x^9$. The parameter $\eta=\pm 1$ reflects the freedom of choosing a spin structure on the cylinder [14].

In the bosonic sector the equations (2.1) are solved by the coherent state

$$|Bp;\mathbf{k}\rangle\!\rangle=\exp\left\{\sum_{n>0}\frac{1}{n}\left(\alpha^i_{-n}\tilde{\alpha}^i_{-n}-\alpha^\mu_{-n}\tilde{\alpha}^\mu_{-n}\right)\right\}|\mathbf{k}\rangle\,. \tag{2.2}$$

In particular, the zero mode equations are solved in the Dirichlet directions by vacua $|B;n_i\rangle$ carrying momenta n_i/R_i and in the Neumann directions by vacua $|B;m_\mu\rangle$ carrying winding $2m_\mu R_\mu$. The states (2.2) are not yet consistent boundary states in the sense of Cardy, since the tree level amplitudes between them do not transform under the S-modular transformation into open string partition functions.

In order to obtain the correct D-brane states it is necessary to localize the branes in the Dirichlet directions and assign definite Wilson lines in the Neumann directions. This is done by Fourier transforming with respect to the positions x_i and the Wilson lines $\tilde{x}_\mu$. The correct boundary state for a Dp-brane sitting at the origin in the external dimensions is therefore given by

$$\begin{aligned}|Bp;\mathbf{x}\rangle = &\int\prod_{l=0}^{5}dk_l|B;k_l\rangle\!\rangle_D\,\times \\ &\times\prod_{j=6}^{6+p}\frac{1}{\sqrt{2R_j}}\sum_{n_j\in\mathbb{Z}}e^{\frac{-in_jx_j}{R_j}}|B;n_j\rangle\!\rangle_D\prod_{\mu=7+p}^{9}\sqrt{R_\mu}\sum_{m_\mu\in\mathbb{Z}}e^{-i2R_\mu m_\mu\tilde{x}_\mu}|B;m_\mu\rangle\!\rangle_N\,,\end{aligned} \tag{2.3}$$

where $\mathbf{x}$ encodes both x_i and $\tilde{x}_\mu$. For completeness and later use, the tree level transition amplitude between two parallel p-branes is then given by

$$\begin{aligned}\mathcal{A}^B_{p-p}(\Delta\mathbf{x}) = &\int_0^\infty\frac{d\tau}{\tau^2}\mathcal{A}^B_{p-p}(\Delta\mathbf{x};\tilde{q})=\int_0^\infty\frac{d\tau}{\tau^2}\int\prod_{l=0}^{5}dk_l\frac{e^{-\frac{\pi}{\tau}k_l^2}}{\eta(\tilde{q})^4}\,\times \\ &\times\frac{1}{\eta(\tilde{q})^4}\prod_{j=6}^{6+p}\frac{1}{2R_j}\sum_{n_j\in\mathbb{Z}}e^{\frac{in_j\Delta x_j}{R_j}}\tilde{q}^{\left(\frac{n_j}{2R_j}\right)^2}\cdot\prod_{\mu=7+p}^{9}R_\mu\sum_{m_\mu\in\mathbb{Z}}e^{2im_\mu R_\mu\Delta\tilde{x}_\mu}\tilde{q}^{(2m_\mu R_\mu)^2}\end{aligned} \tag{2.4}$$

The calculation of the corresponding open string amplitude function is standard and yields

$$\begin{aligned}\mathcal{Z}^B_{op}(\Delta\mathbf{x}) = &\int_0^\infty\frac{d\tau}{\tau}\mathcal{Z}^B_{op}(q;\Delta\mathbf{x})=\int_0^\infty\frac{d\tau}{\tau}\frac{1}{\eta(q)^4}\,\times \\ &\times\frac{1}{\eta(q)^4}\prod_{j=p}^{6+p}\sum_{n_j\in\mathbb{Z}}q^{\frac{1}{2}\left(2R_jn_j+\frac{\Delta x_j}{\pi}\right)^2}\cdot\prod_{\mu=7+p}^{9}\sum_{m_\mu\in\mathbb{Z}}q^{\frac{1}{2}\left(\frac{m_\mu}{R_\mu}+\frac{\Delta\tilde{x}_\mu}{\pi}\right)^2}\,.\end{aligned} \tag{2.5}$$

It can be checked that (2.4) and (2.5) are related by S-modular transformation as required. Observe that the difference $\Delta\mathbf{x}$ the parameters encoding the positions and

Wilson lines leads to a shift of the ground state energy of the different open string sectors as expected.

As in the bosonic case, the fermionic conditions in (2.1) are solved by coherent fermionic states

$$|Fp;\eta\rangle\rangle_{\substack{NSNS\\RR}} = \exp\left\{ i\eta \sum_{r>0} \left(\psi^i_{-r}\tilde{\psi}^i_{-r} - \psi^\mu_{-r}\tilde{\psi}^\mu_{-r} \right) \right\} |F_{g.s.};\eta\rangle_{\substack{NSNS\\RR}} . \tag{2.6}$$

In the NSNS sector $|F_{g.s.}\rangle_{NSNS} = |0\rangle$ is the unique ground state and has no η dependence. In the RR sector, $|F_{g.s.}\rangle_{RR}$ must also solve the zero mode equations in (2.1), and therefore depends non-trivially on η. The different zero mode solutions encode p-brane of all dimension $p = 1, \ldots, 4$ as indicated in (2.1) and ([16]).

The action of the fermionic number operator on the different states (2.6) is given by

$$\begin{aligned} (-1)^F|Fp;\eta\rangle_{NSNS} &= -|Fp;-\eta\rangle_{NSNS} & \qquad (-1)^{\tilde{F}}|Fp;\eta\rangle_{NSNS} &= -|Fp;-\eta\rangle_{NSNS} \\ (-1)^F|Fp;\eta\rangle_{RR} &= (-1)^{7-p}|Fp;-\eta\rangle_{RR} & \qquad (-1)^{\tilde{F}}|Fp;\eta\rangle_{RR} &= |Fp;-\eta\rangle_{RR} . \end{aligned} \tag{2.7}$$

It follows that the GSO invariant boundary states are

$$|Fp\rangle_{NSNS} = \frac{1}{\sqrt{2}} \left(|Fp;+\rangle_{NSNS} - |Fp;-\rangle_{NSNS} \right) \tag{2.8}$$

$$|Fp\rangle_{RR} = \frac{1}{\sqrt{2}} \left(|Fp;+\rangle_{RR} + |Fp;-\rangle_{RR} \right) , \tag{2.9}$$

where p is even in type IIA theory and odd in type IIB.

By world sheet duality, the transition amplitude between the GSO projected boundary states reproduces the NS and R sectors of the open string partition function. The same reasoning also shows that the transition amplitude between two (NSNS) RR boundary states becomes the open string partition function with (no) $(-1)^F$ inserted. An explicit calculation gives the correct supersymmetric p-brane state as

$$|Dp;\mathbf{x}\rangle = |Bp;\mathbf{x}\rangle \otimes \frac{1}{\sqrt{2}} \left(|Fp\rangle_{NSNS} \pm 4i|Fp\rangle_{RR} \right) . \tag{2.10}$$

The RR charges of a D-brane are determined by saturating the boundary state with the corresponding RR vertex operator as in ([16]). By convention, the + sign above is taken to be a brane and the − sign to an anti-brane.

3. Dp-branes on the $\mathbb{Z}_2$ orbifold

In this section D-branes on toroidal orbifolds are studied in the boundary state language. Previous work in a similar direction has appeared in [18, 17], see also [12, 19] for a discussion in the context of non-BPS but stable states.

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From open string considerations it is well known that there are different types of D-branes on orbifolds, depending on the representation of the orbifold group chosen on the Chan-Paton factors [1, 22]. On a $\mathbb{Z}_2$ orbifold there is the regular representation and one-dimensional representations. The resulting D-branes differ physically in their RR-charges: while the regular representation leads to a brane which is only charged under untwisted sector fields, the trivial representation gives D-branes which are charged under both twisted and untwisted sector RR-fields. In the boundary state language this means that the former type of brane contains only Ishibashi states built over untwisted sector primary fields, whereas the latter contains an Ishibashi state built over a twist field. Therefore, we refer to them as "untwisted" and "twisted" branes respectively. In this section we will discuss both untwisted and twisted D0-branes in detail, whereas part of the discussion of the higher dimensional branes is postponed to a later section.

3.1 The untwisted states

The untwisted boundary states are $\mathbb{Z}_2$ invariant combinations of the T^4 boundary states. The orbifold action reverses the sign of both the fermionic and bosonic oscillators in the $a = 6, \ldots, 9$ directions:

$$\alpha_n^a \to -\alpha_n^a \,, \qquad \psi_r^a \to -\psi_r^a \tag{3.1}$$

and the action on the RR vacua given by

$$|F_{g.s.}\rangle_{RR} \to \prod_{a=6}^{9} \sqrt{2}\psi_0^a \prod_{a=6}^{9} \sqrt{2}\tilde{\psi}_0^a |F_{g.s.}\rangle_{RR} \,. \tag{3.2}$$

The RR vacua, on which the even dimensional branes are built, are invariant under this action as can be verified using the conditions (2.1). Moreover, odd dimensional BPS branes in the orbifold directions are projected out as required by the homology of a $K3$ surface. Using this, we see that the full boundary state transforms as

$$|Dp; \mathbf{x}\rangle \to |Dp; -\mathbf{x}\rangle \,, \tag{3.3}$$

leading to the invariant states

$$|Dp; \mathbf{x}\rangle^{Orb} = \frac{1}{\sqrt{2}} \left(|Dp; \mathbf{x}\rangle + |Dp; -\mathbf{x}\rangle \right) . \tag{3.4}$$

The states (3.4) are built only over untwisted vacua and for this reason are charged under the untwisted RR fields as the T^4 states. The computation of the transition amplitudes between two such boundary states is reduced to a computation in the unorbifolded theory. For example the transition amplitude of a boundary state with itself is

$$\begin{aligned}\langle Dp; \mathbf{x}|\tilde{q}^{\frac{1}{2}H_{cl}}|Dp; \mathbf{x}\rangle^{orb} = {} & \frac{1}{2}\langle Dp; \mathbf{x}|\tilde{q}^{\frac{1}{2}H_{cl}}|Dp; \mathbf{x}\rangle^{torus} + \frac{1}{2}\langle Dp; -\mathbf{x}|\tilde{q}^{\frac{1}{2}H_{cl}}|Dp; -\mathbf{x}\rangle^{torus} + \\ & + \frac{1}{2}\langle Dp; \mathbf{x}|\tilde{q}^{\frac{1}{2}H_{cl}}|Dp; -\mathbf{x}\rangle^{torus} + \frac{1}{2}\langle Dp; -\mathbf{x}|\tilde{q}^{\frac{1}{2}H_{cl}}|Dp; \mathbf{x}\rangle^{torus} .\end{aligned} \tag{3.5}$$

The relevant amplitudes for the torus were given in the previous section. As expected, the S-modular transformation of this result leads to an open string partition function with the regular representation on the Chan-Paton factors. The first two terms in this sum are equal to each other and so are the last two. They correspond respectively to strings stretching between a D-brane and itself and strings stretching between a D-brane and its image. If the D-branes are away from the fixed points,[1] only the first two terms can lead to marginal operators, with the corresponding moduli being the position and Wilson line of the brane. At the fixed point $\mathbf{x} = -\mathbf{x}$, there are additional marginal operators coming from the last two terms in the sum (3.5). These arise from external directions to the orbifold, and in the D0-brane world volume theory signal a Coulomb branch opening [1, 22]. The new massless fields arising from the internal directions are gauge equivalent to the marginal operators already in the theory.

The BPS property and the RR charges carried by (3.4) are crucial in correctly identifying these states with spacetime D-branes. The state with $p = 0$ is a D0-brane which can move anywhere inside the orbifold. It carries one unit of D0-brane charge, and has four position moduli. The state with $p = 2$ corresponds to a D2-brane wrapping once one of the six T^2 cycles inherited from the underlying T^4. This assignment again fixes the absolute normalization of the D2-brane charge. Their moduli are equally well understood as transverse position and Wilson line. This can be seen by regarding $T^4/\mathbb{Z}_2$ as an elliptic fibration over a base $\mathbf{P}^1$ with four D_4 singular fibers. The inherited 2-cycles which the D2-branes wrap are elliptic curves with respect to a properly chosen complex structure, and have one complex deformation. These four moduli account respectively for the two $\tilde{x}_\mu$ and two x_i parameters which appear in the boundary state. The state with $p = 4$ is a state carrying only D4-brane charge wrapping the orbifold. We will discuss its physics in the section on T-duality.

3.2 The twisted case

Let us now turn to the twisted boundary states, generalizing the discussion of the bosonic $S^1/\mathbb{Z}_2$ orbifold in [20]. The 16 fixed points of the orbifold are located on points in $\frac{1}{2}W$, where W denotes the winding lattice. Therefore it is natural to label the bosonic twist fields $T_{[\mathbf{v}]}$ by cosets $[\mathbf{v}]$, where $\mathbf{v} \in \frac{1}{2}W$. Each of the 16 twisted vacua $|T_{[\mathbf{v}]}\rangle$ produces an all Dirichlet coherent state

$$|DT_{[\mathbf{v}]}\rangle\rangle = \exp\left\{\sum_{n=1/2}^{\infty} \frac{1}{n}\alpha^i_{-n}\tilde{\alpha}^i_{-n}\right\}|T_{[\mathbf{v}]}\rangle\,. \tag{3.6}$$

Taking into account the fermions, we note that in the twisted NSNS (RR) sector the moding of the internal fermions is (half) integral, and hence there is a unique twisted

[1] The term “fixed point” has been used somewhat loosely, since the parameters for the Neumann directions are Wilson lines.

RR ground state in the internal theory. However, there are fermionic zero modes in the NSNS orbifold directions. The twisted NSNS ground state, which fulfills the zero-mode part of (2.1) must be picked by the same considerations as for the RR ground state on the torus.

The RR Ishibashi state leads to a charge under the corresponding RR vertex operator, implying that the Ishibashi state should correspond to D2-brane wrapped on a collapsed sphere at a fixed point. It is known from the gauge theory analysis that these D2-branes also carry one half unit of D0-brane charge induced by the B-field on the collapsed cycles. In CFT language this is reflected in the requirement of modular invariance. Let us consider open strings starting and ending at the same fixed point with Chan Paton factors transforming in the trivial representation of the $\mathbb{Z}_2$ orbifold group. The open string partition function with the $\mathbb{Z}_2$ generator g inserted reproduces the amplitude between a twisted Ishibashi state and itself. The other part of the open string partition function gives a D0-brane amplitude. The coefficients of the twisted and untwisted parts are fixed by the bosonic open string partition function given by

$$\begin{aligned}\mathcal{Z}^B_{op}(q) &= \mathrm{Tr}_{\mathcal{H}_{op}} \frac{1+g}{2} q^{H^B_{op}} \\ &= \frac{1}{2}\mathcal{Z}^B_{op}(q;0) + \frac{1}{2}\left(q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1+q^n}\right)^4 = \frac{1}{2}\mathcal{Z}^B_{op}(q;0) + \frac{1}{2}\left(\sqrt{\frac{\eta(q)}{\theta_2(q)}}\right)^4 . \end{aligned} \tag{3.7}$$

This shows that in the purely bosonic case the correct fractional D0-brane boundary states are given by

$$|B0B2_{[\mathbf{v}]}\rangle_{\pm} = 2^{-1/2}|B0;\mathbf{v}\rangle \pm 2^{1/2}|BT_{[\mathbf{v}]}\rangle\!\rangle \,. \tag{3.8}$$

It can be checked that these states are mutually consistent among themselves. Note that position of the untwisted part is determined by the twisted part. All other combinations are ruled out by consistency with twisted D4-brane states which we will discuss later. In the full supersymmetric theory there are a-priori eight different states associated to each of the fixed points corresponding to eight possible relative signs. However, only the

$$|D0D2_{[\mathbf{v}]}\rangle_+ \,, \qquad |\overline{D0D2}_{[\mathbf{v}]}\rangle_+ \,, \qquad |\overline{D0}D2_{[\mathbf{v}]}\rangle_- \,, \qquad |D0\overline{D2}_{[\mathbf{v}]}\rangle_- \,. \tag{3.9}$$

have a consistent superstring interpretation [21].

4. T-duality on the orbifold

4.1 The T-duality action on the twist fields

In this section we work out a consistent action of the ‘inherited T-duality’ transformation $\mathcal{T}$ on the twist fields of a $T^d/\mathbb{Z}_2$ orbifold. Similar calculations were done in [27, 24]. We take the underlying T^d to be a rectangular torus with radii R_i and with no B-field turned on.[2] As is well known the spectrum and interactions of the untwisted sector are invariant under $\mathcal{T}$, and we therefore limit our discussion to the twisted sector.

Consider first the bosonic sector of a superstring CFT. The highest weight of a twisted sector is $h_T = \tilde{h}_T = d/16$. Thus, in order to preserve the spectrum, the action of $\mathcal{T}$ can only map twist fields into a linear combination of twist fields. We will now determine the exact map by requiring the three point functions involving twist fields to be invariant. This implies that both the spectrum and interactions are invariant, so that $\mathcal{T}$ is a symmetry of the theory.

On the T^d torus with no B-field the momentum lattice M and the winding lattice W decouple. We denote $\mathbb{Z}_2$-invariant combinations of untwisted sector vertex operators carrying momentum $\mathbf{p}$ and winding $\mathbf{w}$ in these lattices as $V_{\mathbf{p},\mathbf{w}}$. The OPE of two twist fields in a $\mathbb{Z}_2$ orbifold of this torus was given in [25, 26] as follows. The topological selection rule for these amplitudes are very easily seen from the winding lattice.

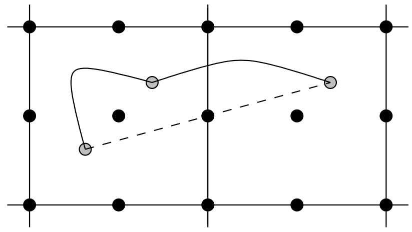

Figure 1: Fusion of a twisted string and a winding string.

The figure above illustrates the joining of a twisted string $T_{[\mathbf{v}]}$ with a winding string $V_{\mathbf{p},\mathbf{w}}$. The result has to be a twisted string around the fixed point $[\mathbf{v} + \frac{1}{2}\mathbf{w}]$. Factorization of the four point amplitude of twist fields gives the exact coefficients in the OPE

$$T_{[\mathbf{v}_1]} T_{[\mathbf{v}_2]} = \sum_{\substack{\mathbf{p}\in M \\ \mathbf{w}\in 2W+2\mathbf{v}_1+2\mathbf{v}_2}} \frac{1}{2}\left((-1)^{2\mathbf{p}\cdot\mathbf{v}_1} + (-1)^{2\mathbf{p}\cdot\mathbf{v}_2}\right) 16^{-\frac{1}{2}(\mathbf{p}^2+\mathbf{w}^2)} V_{\mathbf{p},\mathbf{w}}\,. \tag{4.1}$$

The transformation $\mathcal{T}$ exchanges winding and momentum, and therefore must also act on the twist fields in order to preserve the OPE (4.1). As we show in the appendix, this is achieved by the transformation

$$T_{[\mathbf{v}']} = 2^{-\frac{d}{2}} \sum_{[\mathbf{v}]\in\frac{1}{2}W/W} (-1)^{4\mathbf{v}\cdot\mathbf{v}'} T_{[\mathbf{v}]} \tag{4.2}$$

and therefore $\mathcal{T}$ is a symmetry of the bosonic orbifold CFT. Note that $\mathbf{v}' \in \frac{1}{2}M$, so that the inner product $\mathbf{v}\cdot\mathbf{v}'$ above is well defined.

The supersymmetric case is restricted to $d = 4$ orbifolds by the $\mathbb{Z}_2$ action, but presents no new subtleties. The transformation of the twisted NSNS, RR, NSR or RNS world sheet fermions under $\mathcal{T}$ is identical to that of the untwisted RR, NSNS, RNS or NSR fermions respectively. It follows that $\mathcal{T}$ with the usual action on the untwisted sector and the action (4.2) on the twisted sector is a symmetry of perturbative superstrings compactified on this $T^4/\mathbb{Z}_2$ orbifold.

[2]Note that the B-field on the shrunk 2-cycles does not have to vanish and indeed, as Aspinwall [23] showed the B-field on them is 1/2. This property will be very important in the following sections.

It will prove convenient later to adopt an alternative numbering scheme for the twist fields. There is a natural correspondence between the conjugacy class $[\mathbf{v}]$ of each of the 16 fixed points and four-digit binary numbers b, such that the origin is assigned to 0000 and so forth. We relabel a twist field $T_{[\mathbf{v}]}$ by T_a, where $a = 16 - b$. In this notation the action (4.2) of $\mathcal{T}$ on the twist fields is represented by the 16×16 matrix

$$\mathcal{T}_{a'a} = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \right)^{\otimes 4} . \tag{4.3}$$

4.2 The world volume analysis

In order to compare the previous results with the transformation [6] obtained by world volume analysis, let us briefly recall some relevant facts about type IIA theory compactified on a $T^4/\mathbb{Z}_2$ limit of $K3$. The total integral cohomology of $K3$ forms the lattice $\Gamma_{4,20}$ [30], whose basis vectors are associated to spacetime BPS states. The inner product on this lattice is the oriented intersection number of the corresponding cycles of $K3$. It is natural to work with orthogonal decomposition of this lattice $\Gamma_{4,20} = \Gamma_{3,19} \oplus \Gamma_{1,1}$ corresponding to $H^*(K3,\mathbb{Z}) = H^2(K3,\mathbb{Z}) \oplus H^0(K3,\mathbb{Z}) \oplus H^4(K3,\mathbb{Z})$. In this decomposition, the sub-lattice $\Gamma_{1,1}$ is spanned by vectors ω and ω^* generating $H^4(K3,\mathbb{Z})$ and $H^0(K3,\mathbb{Z})$ respectively over the integers; their intersection form is given by

$$\omega \cdot \omega = \omega^* \cdot \omega^* = 0 , \qquad \omega \cdot \omega^* = 1 . \tag{4.4}$$

Although there exists a basis for $\Gamma_{3,19}$ generating $H^2(K3,\mathbb{Z})$ over the integers, in the orbifold limit it proves more convenient to work with a different one. The elements of this basis are ω_i, ω_i^*, $i = 1,2,3$ corresponding to the 6 inherited two-cycles from the underlying T^4 and γ_a, $a = 1,\dots,16$ representing the 16 collapsed spheres at the fixed points of the $\mathbb{Z}_2$ action. The non-trivial part of their intersection matrix is given by

$$\begin{aligned} \omega_i \cdot \omega_j^* &= 2\delta_{ij} \\ \gamma_a \cdot \gamma_b &= -2\delta_{ab} . \end{aligned} \tag{4.5}$$

The basis (4.5), known as the Kummer basis, generates $H^2(K3,\mathbb{Z})$ over $\mathbf{Q}$, but over the integers only an index two sub-lattice of $\Gamma_{3,19}$ [31]. Observe also that in (4.5) the background B-field needed for a perturbative description of the orbifold takes the form

$$B = -\frac{1}{4} \sum_{a=1}^{16} \gamma_a . \tag{4.6}$$

It is important to keep in mind that the Mukai charges of a point

$$p = Q_0 \omega + Q_2 + Q_4 \omega^* , \tag{4.7}$$

in the $\Gamma_{4,20}$ lattice are not the D0, D2 and D4-brane charges as seen by a low energy observer. The latter are shifted due to the background B-field [32, 33], and are given by

$$\begin{aligned} q_4 &= Q_4 \\ q_2 &= Q_2 - Q_4 B \\ q_0 &= Q_0 + Q_2 \cdot B - \frac{1}{2} Q_4 B^2 \,, \end{aligned} \tag{4.8}$$

where Q_0 already includes the shift due to the $K3$ curvature, i.e. it is the instanton number of the bundle minus its rank.

In [6], a $K3$ duality was formulated, which has the property that it maps the perturbative orbifold $T^4/\mathbb{Z}_2$ to an orbifold of the dual torus $\widetilde{T}^4/\mathbb{Z}_2$. This means in particular that the B-field on the collapsed two-cycles is preserved. The volume of the covering spaces is inverted, meaning that the volume of the orbifold gets mapped to 1/4 of the inverse volume. A theory of D0 and D2-branes on the original orbifold is mapped to the theory of D4-branes and other brane charges on the dual orbifold. Concretely, the gauge theory analysis of the theory of D4-branes wrapping the dual $\widetilde{T}^4/\mathbb{Z}_2$ torus in [6] led to the following transformation of the Kummer lattice generators and ω, ω^*:

$$\begin{aligned} \omega_i &\mapsto \omega_i^{*\prime} \qquad\qquad \omega_i^* \mapsto \omega_i' \qquad\qquad\qquad i = 1,2,3 \\ \gamma_a &\mapsto \omega^{*\prime} + \omega' - \frac{1}{2} D_a \qquad\qquad\qquad\qquad a = 1,\dots,15 \\ \gamma_{16} &\mapsto \omega^{*\prime} - \omega' \\ \omega &\mapsto 2\omega' + 2\omega^{*\prime} - \frac{1}{2}\sum_{a'=1}^{16} \gamma_{a'} \\ \omega^* &\mapsto 2\omega' + 2\omega^{*\prime} - \frac{1}{2}\sum_{a'=1}^{15} \gamma_{a'} + \frac{1}{2}\gamma_{16'} \end{aligned} \tag{4.9}$$

The asymmetry in the transformation of γ_{16} is due to the arbitrary assignment of γ_{16} to the collapsed 2-cycle at the origin. The elements D_a are given by

$$D_a = \sum_{a'=1}^{16} \xi_a^{\ a'} \gamma_{a'} \,, \tag{4.10}$$

where according to the numbering scheme we use $\xi_{a'}^{\ a}$ takes the values 1 or 0 according to

$$\xi_a^{\ a'} = \frac{1}{2}\left(1 - (-1)^{4[\mathbf{v}_{a'}]\cdot[\mathbf{v}_a]}\right). \tag{4.11}$$

4.3 Action on the branes

In the boundary state formalism, the T-duality transformation on the untwisted boundary states is straight-forward. It converts Dirichlet into Neumann boundary conditions and vice versa, so that an untwisted $|Dp;\mathbf{x}\rangle$ state is mapped to an untwisted $|D(4-p);\tilde{\mathbf{x}}\rangle$ state. Here, $\tilde{\mathbf{x}}$ refers to the dual moduli. In particular, the untwisted D0-brane transforms into a state which carries only untwisted D4-brane charge as seen by a low energy observer.

Since we identify the BPS vector ω with the untwisted D0-brane state, the $p = 4$ state should be identified with the transform of ω under (4.9). As evidence for this identification we can compute the RR charges carried by this BPS vector. Using (4.8), its D0 and D2-brane charges vanish as they should. However the world volume analysis leads to twice the fundamental D4-brane charge. This is because under T-duality of the covering space, the D0-brane and its image transform into two D4-branes on the dual covering space. The embedding of the $\mathbb{Z}_2$ action leads to two D4-branes wrapping the dual orbifold with a non-trivial bundle structure encoded in the transformation law (4.9). From the boundary state perspective we will see that this normalization of the D4-brane charge is required by the existence of twisted D4-brane boundary states.

The spacetime BPS state identified with the state (3.9) is the fractional D0-brane of [22]. To compare the boundary states with the world volume analysis, the twist fields are labeled as described before formula (4.3). Inspection of (4.8) shows that the following identifications should be made:

$$\begin{array}{ll}
|D0D2_a\rangle_+ \leftrightarrow \gamma_a & (q_4, q_2, q_0) = \left(0, \gamma_a, \frac{1}{2}\right) \\
|D0\overline{D2}_a\rangle_- \leftrightarrow -\gamma_a + \omega & (q_4, q_2, q_0) = \left(0, -\gamma_a, \frac{1}{2}\right) \\
|\overline{D0}D2_a\rangle_- \leftrightarrow \gamma_a - \omega & (q_4, q_2, q_0) = \left(0, \gamma_a, -\frac{1}{2}\right) \\
|\overline{D0D2}_a\rangle_+ \leftrightarrow -\gamma_a & (q_4, q_2, q_0) = \left(0, -\gamma_a, -\frac{1}{2}\right) .
\end{array}$$

Observe that two of these D2-brane states also carry non-zero flux.

The Neumann state related to the Dirichlet state (3.8) by the duality $\mathcal{T}$ is given by

$$|D4D2_{a'}\rangle_\pm = 2^{-1/2}|D4; \mathbf{v}'\rangle \pm 2^{1/2}|NT_{a'}\rangle\rangle\,, \tag{4.12}$$

where $|NT_{a'}\rangle\rangle$ is the Neumann analog of (3.6) built over the $T_{a'} = \mathcal{T}_{a'a}T_a$ twist fields. Obtaining (4.12) by the $\mathcal{T}$ action on the fractional D0-brane states means that the fixed T^4 Wilson line[3] $\mathbf{v}'$ entering their untwisted parts is again determined by the twist fields $T_{a'}$ in the boundary state. It is important to note though that it can be verified independently of duality considerations that the states (4.12) are consistent boundary states.[4]

Furthermore, we can now fix absolute normalization of D4-brane charges. Being a BPS state, the twisted Neumann state (4.12) must have an integral D4-brane

[3]Note that this is not a Wilson line on the $K3$ surface.

[4]A technical issue alluded to before is whether other combinations of untwisted and twisted boundary states are allowed. This applies both to the fractional D0-branes and D4-branes. As can be verified by open string calculations, consistency of the fractional D0 states with the fractional D4 states excludes other combinations.

charge which we set to unity. Comparison of this state with the inherited state in (3.4) leads to the conclusion that the inherited state indeed carries two units of D4-brane charge, in agreement with the world volume analysis. Note that the $\mathcal{T}$-dual of the fractional D0-brane at the origin carries the charge $\omega^* - \omega$, corresponding to a wrapped D4-brane with trivial gauge bundle.

The agreement with the world volume analysis is easily checked by comparing the action of the CFT $\mathcal{T}$-transformation on the low energy charges with the action of (4.9) on the Mukai charges. Consider for example the BPS vector γ_a, $a \neq 16$. Under (4.9) it transforms into

$$\gamma_a \to \omega^{*\prime} + \omega' - \frac{1}{2} D_a \,, \tag{4.13}$$

whose RR charges are

$$\begin{aligned} q_0' &= 0 \\ q_2' &= -\frac{1}{4}\sum_{a'=1}^{16}\left(1-(-1)^{4[\mathbf{v}_{a'}]\cdot[\mathbf{v}_a]}\right)\gamma_{a'} + \frac{1}{4}\sum_{a'=1}^{16}\gamma_{a'} = \frac{1}{4}\sum_{a'=1}^{16}(-1)^{4[\mathbf{v}_{a'}]\cdot[\mathbf{v}_a]}\gamma_{a'} \\ q_4' &= 1\,. \end{aligned} \tag{4.14}$$

Our CFT analysis reproduces this result. Using (4.3) and the correct normalization for the q_4 charge, the charges of the $\mathcal{T}$ transformed boundary states are in exact agreement with (4.14).

Completing our discussion, we note the existence of consistent fractional D2-brane boundary states which are a combination of four twisted Ishibashi states with an untwisted D2-brane state. The untwisted part corresponds to a D2-brane wrapping once the fixed plane suspended by the four fixed points and carrying half the charge of the untwisted D2-brane which is free to move away from the plane. For each fixed plane there are four such states corresponding to different relative orientations of wrapping the collapsed 2-cycles. It is easy to verify from the discussion above that these states transform properly under $\mathcal{T}$. More precisely, each of the four transformed states corresponds to the new fixed plane intersecting the original one in one common fixed point.

5. Conclusions

When an exact CFT description is possible, boundary states are a powerful tool in deriving results which are otherwise obtained by a more complicated world volume analysis. In this paper we used the boundary state formalism to derive the action (4.9) of the inherited T-duality on the BPS states of type II theory on $T^4/\mathbb{Z}_2$. As it is not easy to find an integral basis of the homology lattice in terms of the orbifold cycles, checking whether (4.9) is an $O(4,20;\mathbb{Z})$ element is difficult. Boundary states are a strong evidence that (4.9) is indeed a T-duality, since the action of the exact perturbative symmetry $\mathcal{T}$ on them coincides with that of (4.9). The inherited

T-duality can be combined with the permutation symmetry of the four orbifold directions to yield a sub-group of $O(4,20;\mathbb{Z})$ preserving the B-field and square T^4. It is interesting to check whether there are additional symmetries which together with $\mathcal{T}$ generates the maximal sub-group of the T-duality group preserving this background.

Acknowledgments

We would like to thank D.E. Diaconescu, B. Fiol, J. Gomis, A. Rajaraman, A. Recknagel, M. Rozali, V. Schomerus and particularly M. Douglas for useful discussions. This work was supported in part by DOE grant DE-FG02-96ER40559.

A. Transformation of the twist fields

Here we show that the transformation

$$T_{[\mathbf{v}']} = 2^{-\frac{d}{2}} \sum_{[\mathbf{v}]\in\frac{1}{2}W/W} (-1)^{4\mathbf{v}\cdot\mathbf{v}'} T_{[\mathbf{v}]} \tag{A.1}$$

preserves the OPE of two twist fields. Combining (A.1) with (4.1) gives

$$T_{[\mathbf{v}'_1]} T_{[\mathbf{v}'_2]} = \frac{1}{2^d} \sum_{\substack{[\mathbf{v}_1],[\mathbf{v}_2] \\ \in\frac{1}{2}W/W}} (-1)^{4\mathbf{v}_1\cdot\mathbf{v}'_1+4\mathbf{v}_2\cdot\mathbf{v}'_2} \sum_{\substack{\mathbf{p}\in M \\ \mathbf{w}\in W+2\mathbf{v}_1+2\mathbf{v}_2}} \frac{(-1)^{2\mathbf{p}\cdot\mathbf{v}_1}+(-1)^{2\mathbf{p}\cdot\mathbf{v}_2}}{2} 16^{-\frac{1}{2}(\mathbf{p}^2+\mathbf{w}^2)} V_{\mathbf{p},\mathbf{w}}\,. \tag{A.2}$$

The first of the two terms gives

$$\begin{aligned} \frac{1}{2^d} \sum_{\substack{[\mathbf{v}_1],[\mathbf{v}_2] \\ \in\frac{1}{2}W/W}} \sum_{\substack{\mathbf{p}\in M \\ \mathbf{w}\in W+2\mathbf{v}_1+2\mathbf{v}_2}} & \frac{1}{2}(-1)^{2\mathbf{p}\cdot\mathbf{v}_1+4\mathbf{v}_1\cdot\mathbf{v}'_1-(2\mathbf{v}_1+\mathbf{w})\cdot 2\mathbf{v}'_2} 16^{-\frac{1}{2}(\mathbf{p}^2+\mathbf{w}^2)} V_{\mathbf{p},\mathbf{w}} = \\ &= \frac{1}{2^d} \sum_{\substack{\mathbf{p}\in M \\ \mathbf{w}\in W}} \frac{1}{2} \sum_{[\mathbf{v}_1]\in\frac{1}{2}W/W} (-1)^{2\mathbf{v}_1\cdot(\mathbf{p}+2\mathbf{v}'_1-2\mathbf{v}'_2)-2\mathbf{w}\cdot\mathbf{v}'_2} 16^{-\frac{1}{2}(\mathbf{p}^2+\mathbf{w}^2)} V_{\mathbf{p},\mathbf{w}} \\ &= \sum_{\substack{\mathbf{p}\in 2M+2\mathbf{v}'_1+2\mathbf{v}'_2 \\ \mathbf{w}\in W}} \frac{1}{2}(-1)^{2\mathbf{w}\cdot\mathbf{v}'_2} 16^{-\frac{1}{2}(\mathbf{p}^2+\mathbf{w}^2)} V_{\mathbf{p},\mathbf{w}}\,. \end{aligned} \tag{A.3}$$

The second term in (A.2) is treated in the same way and leads to the last line above with $\mathbf{v}'_1 \mapsto \mathbf{v}'_2$. Since $V_{\mathbf{p},\mathbf{w}}$ is the vertex operator $V'_{\mathbf{w},\mathbf{p}}$ of the $\mathcal{T}$-transformed theory, this shows that the OPE (4.1) is preserved.

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